ON A MOTION OF A DISK IN A NEWTONIAN FORCE FIELD

(OB ODNOM DVIZHENII DISKA V N'IUTONOVSKOM POLE SIL)

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Iu.A.ARKHANGEL'SKII
(Moscow)

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1. It is known that the problem of the motion of a heavy rigid body, fixed at its center of mass, and situated in a Newtonian force field, was integrated in the general case by Kobb [1] and Kharlamova [2]. However, due to the complexity of the analytic expression of the general solution, the investigation of the motion was carried out only in a few particular cases [2 to 6].

In what follows, the motion is investigated, for one particular case, of the disk

$$A + B = C \tag{1.1}$$

fixed at its center of mass and situated in a Newtonian force field.

With condition (1.1), the equation of motion

$$\frac{dp}{dt} + qr = \alpha\gamma'\gamma'', \qquad \frac{dq}{dt} - pr = -\alpha\gamma\gamma'', \qquad \frac{dr}{dt} = \frac{A-B}{C}(pq - \alpha\gamma\gamma')$$

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma'' \qquad \binom{p\ q\ r}{\gamma\gamma'\gamma''} \qquad \alpha = \frac{3g}{R} \tag{1.2}$$

have a particular solution

$$p = \sqrt{\alpha}\gamma', \qquad q = \sqrt{\alpha}\gamma, \qquad r = 0 \tag{1.3}$$

for which the first four integrals of the system (1.2) degenerate into the following two

$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1, \qquad \gamma\gamma' = \beta \qquad (\beta = \text{const}) \tag{1.4}$$

The dependence of the quantity γ on time is found from Equations

$$d\gamma / d\tau = -\sqrt{f(\gamma^2)}, \qquad f(\gamma^2) = -\gamma^4 + \gamma^2 - \beta^2, \qquad \tau = \sqrt{\alpha}\, t \tag{1.5}$$

the solution of which has the form

$$\gamma = \frac{\text{dn}u}{\lambda}, \qquad u = \beta\mu(\tau - \tau_0), \qquad \lambda^2 = \frac{2}{1 + \sqrt{1 - 4\beta^2}}, \qquad \mu^2 = \frac{2}{1 - \sqrt{1 - 4\beta^2}} \tag{1.6}$$

From Equations (1.4) we get the relations

$$\gamma' = \beta\lambda / \text{dn}u, \qquad \gamma'' = \beta\mu k^2 \,\text{sn}u\, \text{cn}u / \text{dn}u, \quad k^2 = 1 - \lambda^2 / \mu^2 \tag{1.7}$$

which together with Equations (1.6) and (1.3) give the full solution of the problem in the case under consideration. It is also assumed that

$$0<\beta<1/2$$

and, in the chosen movable system of coordinates, thr following conditions are satisfied at the initial time $\gamma>0,\ \gamma'>0$.

2. For the investigation of the motion we will follow the displacement in space of the x-axis of the movable system of coordinates. Then, Euler's angles θ, φ, and ψ will be obtained, in our case, from the corresponding equations [7] in the form

$$\cos\theta=\gamma,\qquad d\psi\,/\,d\tau=\beta\,/\,(1-\gamma^2),\qquad \tan\varphi=\gamma'\,/\,\gamma'' \tag{2.1}$$

Substituting into (2.1) in place of γ, γ' and γ'' their Expressions (1.6) and (1.7) and integrating (which reduces to the evaluation of a normal elliptic integral of the third kind), we cab obtain the dependence of Euler's angles on time.

We shall, however, conduct the analysis of the motion without using the explicit relation of Euler's angles with time.

From Equations (1.5), (1.6) and (2.1) it follows, that, on a stationary unit sphere with its center at the center of mass of the disk, the trace of the x-axis will describe a curve between the paralleles $\gamma_1=\lambda^{-1}$ and $\gamma_2=\mu^{-1}$ on the upper hemisphere, satisfying the inequality

$$\gamma_2<1\,/\,\sqrt{2}<\gamma_1 \tag{2.2}$$

Let V be the angle, formed by the tangent to the trajectory of the x-axis on the unit sphere with the meridian. Then, from Equation [8]

$$\tan V=\beta\,/\,\sqrt{f(\gamma^2)} \tag{2.3}$$

it follows that the angle V becomes a right angle each time it takes on the values γ_1 and γ_2 and, therefore, the trajectory of the x-axis is tangent to both parallels. Since $\tan V$ does not become zero, the trajectory does not describe a loop, does not possess turning points and represents a smooth curve, alternately tangent to the upper and lower parallels.

The ratio of the period of precession $T_\psi=2\pi\,/\,\langle d\psi\,/\,d\tau\rangle$, where

$$\left\langle\frac{d\psi}{d\tau}\right\rangle=\frac{1}{2}\left[\frac{d\psi}{d\tau}(\gamma_1)+\frac{d\psi}{d\tau}(\gamma_2)\right]=\frac{1}{2\beta}$$

is the mean speed of precession [5], to the period of nutation

$$T_\gamma=2K(k^2)\,/\beta\mu,$$

where $K(k^2)$ is the complete elliptic integral of the first kind, will be

$$T_\psi\,/\,T_\gamma=2\pi\beta^2\mu\,/\,K(k^2) \tag{2.4}$$

From Equations

$$\tan\varphi=\tan V,\qquad d\varphi\,/\,d\tau=2\beta\,(1/2-\gamma^2)\,[\gamma\,(1-\gamma^2)]^{-1} \tag{2.5}$$

derived on the basis of (1.4),(2.1) and (2.3), it follows that according to φ the disk will have a periodic motion, the velocity of which changes sign upon the passing of x-axis through the parallel $\gamma=1/\sqrt{2}$ and the magnitude of the angle φ changes in the range

$$\varphi_1\leqslant\varphi\leqslant\pi-\varphi_1,\qquad \tan\varphi_1=\beta\,(1/4-\beta^2)^{-1/2} \tag{2.6}$$

3. Let us investigate the limiting cases $\beta=0$ and $\beta=\frac{1}{2}$. When $\beta=0$, from (1.4) and (1.5) we get the Expressions

$$\gamma=\operatorname{sch}(\tau-\tau_0),\qquad \gamma'=0,\qquad \gamma''=\tanh(\tau-\tau_0) \tag{3.1}$$

which, when substituted into Equations (2.1), give

$$\cos\theta = \operatorname{sch}(\tau - \tau_0), \qquad \psi = \psi_0, \qquad \varphi = 0 \tag{3.2}$$

Thus, the motion in this case reduces to the turning of the disk around the line of the nodes, during which a right angle with the direction, joining the center of attraction with the fixed point, is reached after an infinite time. (Note that the motion is a particular case of the motion of a physical pendulum [6]). When $\beta = \frac{1}{2}$

$$f(\gamma^2) = -(1/2 - \gamma^2)^2 \tag{3.3}$$

and from (1.4), (1.5) and (2.1) we get

$$\gamma = \gamma' = 1/\sqrt{2}, \quad \gamma''=0; \qquad \theta = \varphi = 1/4\,\pi, \qquad \psi - \psi_0 = \tau \tag{3.4}$$

It thus follows that, in the case under consideration, the disk will perform a uniform rotation about an axis, passing through the fixed point and the center of attraction.

We note, in conclusion, that the solution obtained in (1.3), (1.6) and (1.7) does not have its analogy in Euler's classical case, and all the results in the motion and disk considered are only subject to the presence of a Newtonian force field.

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